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Asymptotic Properties of a Simple Random Motion

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A random walker in $\mathbb{R}^N$ is considered. At each step the walker picks a point in $\mathbb{R}^N$ from a fixed finite set of destination points. Having chosen the point, the walker moves a fraction r ($r<1$) of the distance toward the point along a straight line. Assuming that the successive destination points are chosen independently, it is shown that the asymptotic distribution of the walker's position has the same mean as the destination point distribution. An estimate is obtained for the fraction of time the walker stays within a ball centered at the mean value for almost every destination sequence. Examples show that the asymptotic distribution could have intricate structure.

KEY WORDS: Random walk; fractals.

1. INTRODUCTION

Consider an indecisive walker in $\mathbb{R}^N$ who wants to reach one of the destinations contained in a fixed finite destination set, $D=\{d_1, d_2,..., d_k\}$. The walker starts from a point x_0 and chooses a destination point d_j with probability p_j. Having chosen d_j, it walks from x_0 toward d_j along the direction of d_j-x_0 a distance $r\,|d_j-x_0|$, where $0<r<1$ is a constant. Once it reaches $x_0+r(d_j-x_0)$, the walker stops and picks a destination from D independently of the previous choice, and repeats the walk. In this paper I study some of the asymptotic properties of the position of the random walker described above. This problem arose in the study of a neural network model,[1] though the problem is interesting on its own as a problem in probability theory.

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2. THE MODEL

Let us denote the position of the walker after the nth step by X_n and the destination random vector chosen before the nth step by Y_n. Assume that $\{Y_n\}_1^\infty$ are mutually independent, with a common probability mass distribution $\pi = \{p_1, p_2, ..., p_k\}$, $\sum_1^k p_i = 1$. From the description of the walk it is clear that X_n satisfies the following recursion relation:

$$X_n = X_{n-1} + r(Y_n - X_{n-1}) = (1-r)\, X_{n-1} + rY_n \tag{1}$$

$$= (1-r)^n\, x_0 + \sum_{j=0}^{n-1} r(1-r)^j\, Y_{n-j} \tag{2}$$

Denote the probability distribution of X_n (supported on a finite set in $\mathbb{R}^N$) by μ_n. Let $\{Y'_n\}$ be a sequence of i.i.d. random vectors with the same common distributions (π) as $\{Y_n\}$. Consider the random series $\sum_{j=1}^\infty r(1-r)^{j-1}\, Y'_j$. From the boundedness of Y_1 it easily follows that this series converges for *every* choice of destination sequence, to a random vector Z. Let us denote the distribution of Z by ν.

Lemma 1. X_n converges in distribution to Z.

Proof. Let

$$X'_n = (1-r)^n\, x_0 + \sum_1^n r(1-r)^{j-1}\, Y'_j \tag{3}$$

Let μ'_n be the distribution of X'_n. Since $\{Y_n\}$ and $\{Y'_n\}$ are i.i.d. random vectors with the same common distribution, $\mu_n = \mu'_n$. Since $(1-r)^n\, x_0$ converges to zero as n tends to ∞, X'_n converges to Z for every choice of destination sequence. This implies $\mu'_n \Rightarrow \nu$, since $\mu_n = \mu'_n$, we have $\mu_n \Rightarrow \nu$.

Remark. Hereafter, unless specified otherwise, almost surely will mean for almost every destination sequence with respect to the infinite product of π. Note that while X'_n converges for all destination sequences, X_n diverges almost surely. (Easily proved by using the Borel–Cantelli lemma.)

3. ORBIT STRUCTURE: ASYMPTOTIC PROPERTIES

We have seen that μ_n converges in distribution to ν. From the definition of ν it is clear that ν is supported on the convex hull C of $\{d_1, d_2, ..., d_k\}$. It is clear from Eq. (1) that if the walker starts in the convex hull, it stays there. If C has nonzero volume in $\mathbb{R}^N$, then it can be shown that starting from anywhere in $\mathbb{R}^N$, after a finite number of steps the walker will be in C (almost surely). If C is a lower-dimensional manifold in $\mathbb{R}^N$, then it can be seen that the distance from X_n to C converges to zero, almost surely. Given these observations, for simplicity I assume that the walk starts from inside C. I now compute the mean and dispersion of Z,

$$E(Z) = \sum_{j=1}^{\infty} r(1-r)^{j-1} E(Y_j') = E(Y_1') = \bar{Z}$$

Thus, the mean value of v is same as that of the destination vector Y_1. Here $E(|Z-\bar{Z}|^2)$ measures the dispersion of v about its mean $\bar{Z}$,

$$E(|Z-\bar{Z}|^2) = \sum_1^N E(Z^{(k)} - \bar{Z}^{(k)})^2 = \sum_1^N \operatorname{Var}(Z^{(k)})$$

where the random variable $Z^{(k)}$ is the kth component of Z,

$$\operatorname{Var}(Z^{(k)}) = \operatorname{Var}\left(\sum_1^{\infty} r(1-r)^{j-1} Y_j'^{(k)}\right) = \sum_1^{\infty} \operatorname{Var}(r(1-r)^{j-1} Y_j'^{(k)})$$

Since $\{Y_j'^{(k)}\}$ are independent, this is equal to

$$\sum_1^{\infty} r^2(1-r)^{2(j-1)} \operatorname{Var}(Y_i'^{(k)}) = \frac{r}{2-r} \operatorname{Var}(Y_i'^{(k)})$$

Therefore,

$$E(|Z-\bar{Z}|^2) = \frac{r}{2-r} \sum_{k=1}^{N} \operatorname{Var}(Y_i'^{(k)}) = Br \tag{4}$$

where

$$B = \sum_1^N \frac{\operatorname{Var}(Y_i'^{(k)})}{2-r}$$

It follows from the Markov inequality that

$$P(|Z-\bar{Z}| > \alpha) \leqslant \frac{E(|Z-\bar{Z}|^2)}{\alpha^2} \tag{5}$$

for all $\alpha > 0$.

I now estimate the mean fraction of time (steps) the walker spends outside a ball of radius α centered at $\bar{Z}$.

Lemma 2. Let $A_\alpha = \{x \in \mathbb{R}^N \mid |x-\bar{Z}| \geqslant \alpha\}$:

$$\lim_{K\to\infty} E\left(\frac{1}{K}\sum_{n=1}^{K} \chi_{A_\alpha}(X_n)\right) = \lim_{K\to\infty} \frac{1}{K}\sum_1^K P(|X_n - \bar{Z}| \geqslant \alpha) \leqslant \frac{Br}{\alpha^2} \tag{6}$$

Proof.

$$P(|X_n - \bar{Z}| \geqslant \alpha) = P(|X'_n - \bar{Z}| \geqslant \alpha)$$

$X'_n \xrightarrow[n \to \infty]{} Z$ almost surely. Therefore,

$$P(|X_n - \bar{Z}| \geqslant \alpha) = P(|X'_n - \bar{Z}| \geqslant \alpha) \xrightarrow[n \to \infty]{} P(|Z - \bar{Z}| \geqslant \alpha)$$

Thus, we have

$$\lim_{K \to \infty} \frac{1}{K} \sum_1^K P(|X_n - \bar{Z}| \geqslant \alpha) = \lim_{K \to \infty} P(|X_n - \bar{Z}| \geqslant \alpha) = P(|Z - \bar{Z}| \geqslant \alpha) \leqslant \frac{Br}{\alpha^2}$$

Having obtained an estimate of the mean fraction of time the walker spends outside a ball centered at $\bar{Z}$, I now obtain an estimate for the same quantity, pointwise, i.e., for almost every destination sequence. While the next result implies Lemma 2, the proof of Lemma 2 given above is much simpler than the proof of the next result. This is the reason for proving Lemma 2 separately.

4. PROPERTIES OF MARKOV CHAIN $\{X_n\}$

From the independence of $\{Y_n\}$ it easily follows that the sequence $\{X_n\}$ is a Markov chain with state space C. Denote the transition probability kernel for $\{X_n\}$ by $P(x, \cdot)$. Since X_n is a function of $\{Y_i\}_1^n$ and x_0, $\{X_n\}$ can be defined on the destination sequence probability space (D_∞, π_∞), where

$$D_\infty = \bigtimes_{i=1}^{\infty} D_i, \quad D_i = D, \qquad \pi_\infty = \bigtimes_1^{\infty} \pi_i, \quad \pi_i = \pi$$

I show that v is a stationary initial distribution for $\{X_n\}$. Let X_0 be a random vector in C with a distribution v. Observe that the distribution of $\sum_1^\infty r(1-r)^{j-1} Y_{j+1}$ is v. Therefore the distribution of $X_1 = (1-r) X_0 + rY_1$ is same as the distribution of

$$(1-r) \sum_1^\infty r(1-r)^{j-1} Y_{j+1} + rY_1$$

which is v. Let (Ω, Q) be a probability space on which the initial random vector X_0 is defined. The Markov chain $\{X_n\}$ with an initial random vector $X_0 = W$, denoted by $\{X_n(W)\}$, can be defined on $(\Omega, Q) \times (D_\infty, \pi_\infty)$. It is clear from the definition of X_n that there is a coupling of $\{X_n(W_1)\}$ and $\{X_n(W_2)\}$ on $(\Omega, Q) \times (D_\infty, \pi_\infty)$ satisfying

$$X_n(W_1) - X_n(W_2) = (1-r)^n (W_1 - W_2) \tag{7}$$

where W_1 and W_2 are random vectors on Ω, taking values in C. Let $\{\hat{X}_n\}$ denote the Markov chain with initial distribution ν.

Theorem. $\{\hat{X}_n\}$ is an ergodic Markov chain.

Proof. Suppose C_1 and C_2 are two invariant subsets of C, with $\nu(C_1)>0$, $\nu(C_2)>0$. If one restricts ν to C_1 and C_2, one obtains two measures ν_1 and ν_2 for $\{X_n\}$. Let $\{X_n^i\}$ be the stationary Markov chain with an initial random vector X_0^i distributed according to ν_i ($i=1$ or 2). Then from (7)

$$X_n^1 - X_n^2 = (1-r)^n (X_0^1 - X_0^2)$$

which implies that $X_n^1 - X_n^2$ converges to zero $(Q \times \pi_\infty)$ a.e. Since the distribution of X_n^i is ν_i for all $n \in IN$, and a.e. convergence implies convergence in distribution, one has $\nu_1 = \nu_2$, from which one gets $\nu(C_1 \Delta C_2)=0$. This proves the theorem.

Let f be a continuous real-valued function on C. From the ergodic theorem one has

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\hat{X}(\omega)) = \int f\, d\nu \tag{8}$$

for $(Q \times \pi_\infty)$ a.e. ω. If one denotes the Markov chain starting from x by $\{X_n(x)\}$, then (7) and continuity of f imply that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(X_n(x)) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\hat{X}) = \int f\, d\nu \tag{9}$$

for all $x \in C$. Using this result and by considering a sequence of continuous functions approximating the characteristic function of a sphere in C, one can easily obtain a pointwise version of Lemma 2.

Remarks. (1) In general one cannot extend (9) to $f \in L_1(\nu)$. This is because ν could be a measure which is singular with respect to Lebesgue measure, as is shown in the next section. If $r=2/3$ and $Y_n=0$ or 1, ν is supported on a subset of the Cantor ternary set. Let $S \subset C=[0,1]$ be the support of ν. If $f=\chi_S$, and we take $x_0=x$, where x is not an element of the Cantor set, it is easy to see from (2) that $f(X_n(x))=0$ for all n. This shows that (9) need not hold for all $x \in C$ if we only assume $f \in L_1(\nu)$. Of course, (8) is true for all $f \in L_1(\nu)$, from which it follows that (9) is true for $f \in L_1(\nu)$ and ν–a.e. $x \in C$.

(2) The arguments given here can be easily extended to the case where the destination point random variables $\{Y_n\}$ are i.i.d. with a general distribution π (with some assumptions about the existence of moments).

5. SUPPORT OF ν

It has been shown that μ_n converges in distribution to ν which is the distribution of $Z=\sum_1^\infty r(1-r)^{j-1} Y_j$. The measure ν is also a stationary measure for the Markov chain $\{X_n\}$ of the random walker's position. I show that the support of ν can be quite intricate by considering some examples. It is not hard to see that ν is not a discrete measure. Therefore the question of interest concerns the absolute continuity of ν with respect to Lebesgue measure. I show that there exist a large range of parameter (r) values for which ν is supported on a subset of a fractal. Consider the random motion in $[0, 1]$, where the set $D=\{0, 1\}$. If $r=1/2$ and $p_1=p_0=1/2$,

$$Z=\sum_1^\infty r(1-r)^{j-1} Y_j=\sum_1^\infty \left(\frac{1}{2}\right)^j Y_j$$

This is just the binary expansion for real numbers in $[0, 1]$. It is well known that ν is equivalent to the Lebesgue measure.[2] If $p_0 \neq 1/2$, then ν is a measure on $[0, 1]$ singular with respect to Lebesgue measure.[3] If $r \neq 1/2$, then let us try to locate the invariant measure for $\{X_n\}$. One can think of the motion of a random walker starting anywhere as being pulled toward either 0 or 1. If $r > 1/2$, then after one step the whole interval $[0, 1]$ gets "attracted" into the set $[0, 1-r] \cup [r, 1]$. After the first step the walker can never enter the middle $(2r-1)$ portion of the interval. Repeating this argument, one concludes that the support of ν is a subset of the middle $2r-1$ Cantor set. If $r=2/3$, one gets the Cantor ternary set. If $r<1/2$, this argument does not apply. In this case the nature of ν depends more on the probabilistic structure through the choices of p_0 and p_1. If one considers more than two destinations in one dimension, that is, one looks at $0=d_1<d_2< \cdots <d_K=1$, then if $r>1-1/K$, a similar argument will show that ν is supported on a subset of a singular set. The structure of the singular set is slightly more complicated due to possible overlap of intervals $(rd_i, rd_i+(1-r))$ and $(rd_{i+1}, rd_{i+1}+(1-r))$.

The same argument can be used in higher dimensions to see that ν is supported on a subset of a fractal for a range of parameter values. I illustrate this by considering three destination points in the plane. The points d_1, d_2, and d_3 form an equilateral triangle (see Fig. 1). After the first step if the walker chose d_j, it ends up in the smaller similar triangle T_j. Successive steps restrict the walker's position to smaller and smaller triangles as shown in the figure. When $r=1/2$, ν is supported on a subset of the Sierpinski gasket.[4] One could obtain other fractals as the possible support of ν by choosing the destination points d_j and the contraction parameter r appropriately.

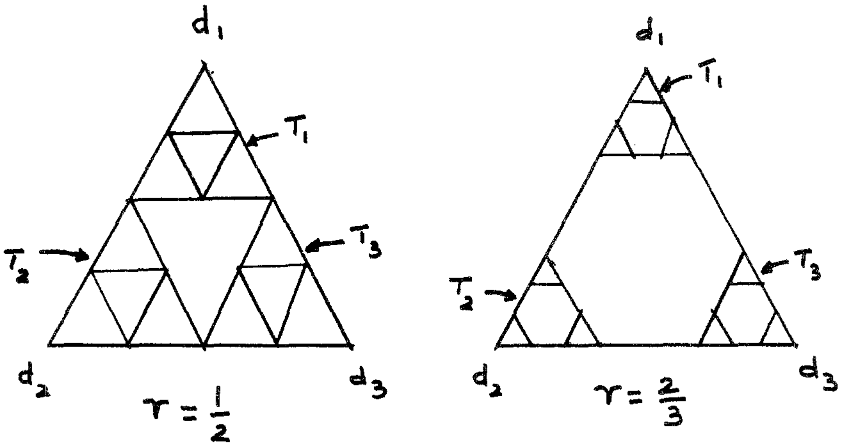

Fig. 1. Random walk with three destinations.

Some interesting open questions concern the nature of ν when $r < 1/2$ in one dimension and also the nature of the subset of the fractal sets which support ν when r is in the appropriate range.

6. CONCLUDING REMARKS

Another way of stating the result $\mu_n \Rightarrow \nu$, when ν is supported on a fractal set, is that the random motion gets attracted to a random walk on a fractal. In the language of dynamical systems, we have obtained a class of "strange" attractors for a random motion. The random walk described in this paper might be useful in generating fractal sets or subsets of fractal sets.

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REFERENCES

1. D. Clark and R. Halpern, Grossberg learning in linear threshold models, Preprint.
2. L. Breiman, *Probability* (Addison-Wesley, Reading, Massachusetts, 1968).
3. P. Billingsley, *Probability and Measure* (Wiley, New York, 1986).
4. B. Mandelbrot, *Fractal Geometry of Nature* (W. H. Freeman, 1983).

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